Cream Pie Fight

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Cream pie fight is sometimes senseless. People watching this game on television or movie always laugh until their bellies hurt while actors are being hit by cream pies thrown by others. With some serious thought; however, cream pie fight may be a good learning topic in mathematics. Here are some discoveries for sharing.

Theorem 1 Odd Pie Fight Theorem

Suppose that there is odd number of people standing in a room, their mutual distances are distinct. Each person has only one cream pie to throw to his nearest neighbour. Then there is at least a survivor.

Let me specify some points on the theorem :

(1) The room is a plane and people are denoted by points,

(2) There are at least 3 people in the room,

(3) " Distinct mutual distances " means that if two people are standing ***d*** units from one another, then no other people are ***d*** units from one another, in other words, each person has a unique neighbour.

(4) Each throw should aim at one's nearest neighbour and hits perfectly the unlucky person.

It is interesting that Odd Pie Fight Theorem can be proved by Mathematical Induction !

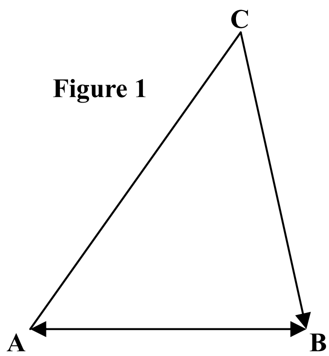
For simplicity, we use $A \to B$ to denote the event that A throws at B and also $A \leftrightarrow B$ to denote that A throws at B and B throws at A.

Proof

Let P(n) be the proposition :

"There is always a survivor in every $2n - 1$ $(n \geq 2)$ people in the pie fight".

Figure 1



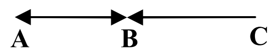
For P(2),

(a) Suppose there are three people, A, B and C standing in a scalene triangle as in Figure 1.
Without loss of generality, we suppose $AB < BC < CA$.
Then $A \leftrightarrow B$, $C \to B$ and C is the survivor.

(b) If the three people are standing on a line as in Figure 2.
Then $A \leftrightarrow B$, $C \to B$ and C is the survivor.

$\therefore$ P(2) is true.

Figure 2



Suppose that P(k) is true for some $k \in \mathbb{N} \setminus \{1\}$,
that is, there is a survivor in every $2k - 1$ ($k \geq 2$) people in the pie fight.

For P(k + 1),
Now we have $2(k + 1) - 1 = 2k + 1$ people throwing pies at each other.
Among all distances between the 2k + 1 people, there is the shortest distance ***d***.
Let A and B be the people standing with this unique minimal distance ***d*** apart.

Case 1 If there is nobody throwing pie at A or B as in Figure 3, then A↔B. If we ignore A and B, we get 2k – 1 people. By P(k), there exists a survivor in these 2k – 1 people.

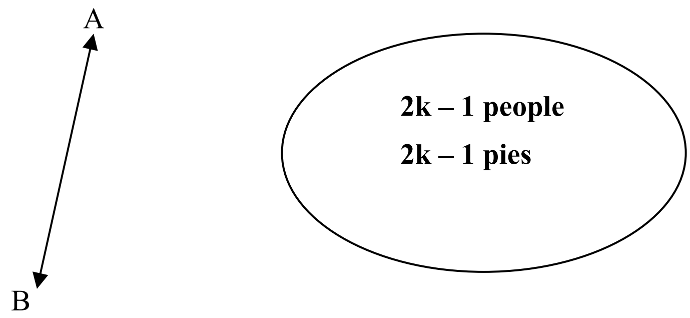


Figure 3

Case 2 If there is someone else throwing pie at A or B as in Figure 4, we still have A↔B. Here we cannot use our inductive hypothesis P(k). However, since A and B together receive at least three pies, the other 2k – 1 people receive at most 2k – 2 pies. Therefore there is not enough pie to go around and there is at least one survivor.

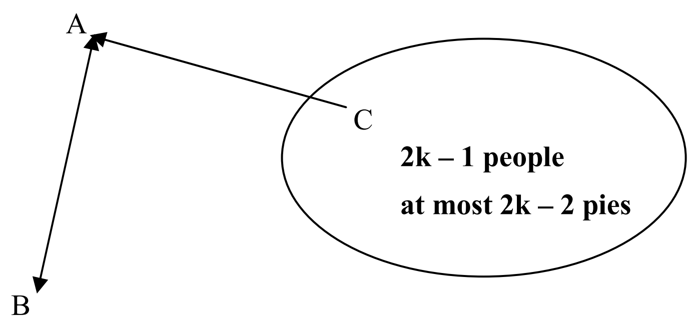


Figure 4

∴ P(k + 1) is true and the Mathematical Induction completes.

We can see that the **Odd Pie Fight Theorem** cannot apply to even number of people. A counter example is that : $X_1 \leftrightarrow X_2$, $X_3 \leftrightarrow X_4$, $X_5 \leftrightarrow X_6$, … , $X_{2n-1} \leftrightarrow X_{2n}$, where the distances within pairs are smaller than the distance across pairs.

Now, we define a **victim** to be a person who cannot survive after pie fight and a very unlucky **super-victim** with **multiplicity k** to be a person who is hit by k pies, where $k > 1$.

Theorem 2 Super-victim Theorem

(a) If there is one (or more) super-victim in pie fight, then there is at least a survivor.

(b) If there is one (or more) super-victim of multiplicity k, there are at least k – 1 survivors.

Proof

(a) Suppose that we start with n people and there is, for simplicity, a super-victim A. We then take away this super-victim together with the pies hitting him. For the rest of people, n – 1 of them, are being attacked by less then n – 2 pies, that is, one pie thrown by the super-victim together with less than n – 3 pies not throwing towards the super-victim. The final result is that we have n – 1 people and at most n – 2 pies attacking them. Therefore there are not enough pies to go around and there is at least one survivor.

(b) This is a more elaborate statement of **(a)** , and the proof is similar.

Theorem 3 Multiplicity Theorem

In a pie fight, there is no super-victim with multiplicity more than five.

Proof

As in Figure 5, let X be a super-victim.

Suppose that A and B both throw at X.

Since their mutual distances are distinct, we have:

$AX < AB$ and $BX < AB$.

Then AB is the longest side in ΔABC.

Since greater side opposite greater angle, ∠AXB is greatest.

We therefore have $\angle AXB > 60^o$.

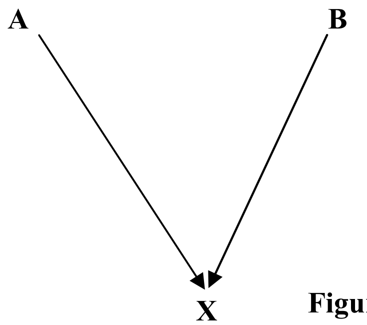


Figure 5

Now, since the sum of angles at a point is 360^o ,

we can at most arrange 5 people around X to attack X.

The most unlucky person is hit by at most 5 pies!

Theorem 4 Cross Path Theorem

The paths of pies do not cross in any pie fight.

Proof

Let us suppose A→B and C→D and their path intersects at M as shown in Figure 6.

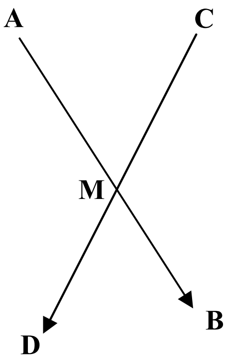


Figure 6

Since their mutual distances are distinct, we have:

$AB < AD$ and $CD < CB$

Adding, we get $AB + CD < AD + CB$ …. **(1)**

On the other hand, by triangular inequality, we have,

$AD < AM + MD$ and $CB < CM + MB$

Adding, we get $AD + CB < AM + MB + CM + MD = AB + CD$ …. **(2)**

Obviously **(1)** contradicts with **(2)** and we cannot have cross path.

Theorem 5 Closed Polygon Path Theorem

The paths of any pie fight cannot form a closed polygon.

Proof

Suppose $X_1 \to X_2, X_2 \to X_3, , \ldots, X_{n-1} \to X_n$ and $X_n \to X_1$, so that we have a closed polygon .

Since X_1 throws at X_2 and not X_n , we have

$X_1X_n > X_1X_2$ …. **(3)**

On the other hand, if we investigate $X_2, X_3, \ldots, X_n$ in turn, we get:

$X_2X_3 < X_1X_2$, $X_3X_4 < X_2X_3$, …, $X_nX_1 < X_{n-1}X_n$.

Combining these inequalities, we get $X_1X_2 > X_2X_3 > X_3X_4 > \ldots > X_nX_1$ …. **(4)**

Obviously **(3)** contradicts with **(4)** and we cannot have closed polygon path.

A small challenge

Do you think that the following theorem is correct? If yes, prove it. If no, disprove it by using a counter-example.

Maximal Survivors Theorem

Let n ($n \geq 3$) be the number of people and $f(n)$ be the maximum number of survivors after the cream pie fight. Then $f(n) = n - 2$.

Hints:

- **(a)** Apply **Theorem 3 (Multiplicity Theorem)** .
- **(b)** Find $f(3), f(4), f(5), \ldots, f(11)$.

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